Full well-posedness of point vortex dynamics corresponding to stochastic 2D Euler equations

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Abstract

The motion of a finite number of point vortices on a two-dimensional periodic domain is considered. In the deterministic case it is known to be well posed only for almost every initial configuration. Coalescence of vortices may occur for certain initial conditions. We prove that when a *generic* stochastic perturbation compatible with the Eulerian description is introduced, the point vortex motion becomes well posed for every initial configuration, in particular coalescence disappears.

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1 Introduction

Existence and uniqueness questions for the 2D Euler equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad u|_{t=0} = u_0 \tag{1}$$

are well understood in suitable functions spaces (see, for instance, [14] and [13] for a review of several results). One of the classical results is the existence of solutions when u_0 is in the Sobolev space $W^{1,2}$ and the uniqueness when the (scalar) vorticity $\xi = \nabla^{\perp} \cdot u = \partial_2 u_1 - \partial_1 u_2$ is bounded.

The case when the vorticity is a signed measure received also a lot of attention, due to the interest in the evolution of vortex structures like sheets or points of vorticity concentration. See [14] for a review. Deep existence and stability results for distributional vorticities which do not change sign have been proved, first for a class of distributions which includes vortex sheets but not vortex points, then also for point vortices (see among others [4], [18]).

Uniqueness is an open problem in all such cases. When the vorticity has variable sign and is, for instance, pointwise distributed, even a reasonable formulation of the Euler equations is missing. However, in the case of point vortices, there are good reasons to replace the Eulerian formulation with a Lagrangian one, based on the autonomous motion of a finite number of point vortices.

The Lagrangian formulation of point vortex motion gives rise to a finite dimensional ordinary differential equation, which is well posed only for almost all initial configurations with respect to Lebesgue measure. One can give explicit examples of initial configurations such that different vortices coalesce in finite time. In such a case the Lagrangian equations loose meaning. Perhaps a proper Eulerian description could be meaningful also after the coalescence time, but a rigorous formulation of this fact is not known.

The purpose of this paper is to show that the previous pathology, namely the existence of initial configurations which coalesce in finite time, is prevented by the presence of suitable noise in the system. The point vortex motion is well defined for all times and all initial configurations, under suitable noise perturbations (which may be arbitrarily small). Let us describe our aim in more detail.

As shortly recalled in the next section, Euler equations can be recast in terms of vorticity as the system

$$\frac{\partial \xi}{\partial t} + u \cdot \nabla \xi = 0, \quad \xi|_{t=0} = \xi_0, \tag{2}$$

$$u = -\nabla^{\perp} \Delta^{-1} \xi. \tag{3}$$

Concepts of weak solutions of the Euler equations are meaningful even for distributional vorticity ξ, when u is sufficiently regular; square integrable is sufficient, see the theory of vortex sheet solutions, where there are at least some existence theorems (see [14]). The limit case when ξ is the sum of finite number of delta Dirac masses

$$\xi(.,t) = \sum_{i=1}^{n} \omega_i \delta_{x_t^i} \tag{4}$$

is unfortunately too singular: the velocity field is not square integrable (so even the weakest form of (1) is not meaningful) and its singularity coincides with the delta Dirac points of the vorticity (so also (2) is not meaningful). In spite of this, there are good arguments, based on the limit of regular solutions supported around the ideal point vortices [15], to accept that a certain finite dimensional differential equation for the position of the point vortices is the correct physical description of the evolution of ξ in (4). The equations for the evolution of the positions of point vortices have the form

$$\frac{dx_t^i}{dt} = \sum_{j \neq i} \omega_j K(x_t^i - x_t^j), \quad i = 1, ..., n. \tag{5}$$

A few more details are explained in the next section (see also [16]).

If we call $X_0 = (x_0^1, ..., x_0^n)$ the initial condition in $\mathbb{R}^{2n}$ of the system of n point vortices, a result of existence and uniqueness for *Lebesgue almost every* X_0 is known, see [15]. But there are examples of initial condition X_0 such that vortices collide in finite time and a global solution does not exist.

The purpose of this research is to investigate the effect of a multiplicative noise on the Euler equations, in the form

$$d\xi + u \cdot \nabla \xi dt + \sum_{k=1}^{N} \sigma_k(x) \cdot \nabla \xi \circ d\beta_t^k = 0, \quad \xi|_{t=0} = \xi_0, \tag{6}$$

where $\sigma_k(x)$ are suitable 2d vector fields and $\{\beta_t^k\}_{k=1,...,N}$ are independent Brownian motions. Note that u is again reconstructed from ξ by means of Biot-Savart law (3). Linear transport equations are regularized by multiplicative noise, see [6]: non-uniqueness phenomena of the deterministic case disappear under the random perturbation. Our aim, in principle, is to prove a similar regularizing effect for the nonlinear problem (6). However this is a very difficult problem and at the moment we are not able to solve it. The result we present in this work is in some sense a first step and concerns the stochastic point vortex dynamics which corresponds to equation (6) and has the form

$$dx_t^i = \sum_{j \neq i} \omega_j K(x_t^i - x_t^j) dt + \sum_{k=1}^{N} \sigma_k(x_t^i) \circ d\beta_t^k, \quad i = 1, ..., n. \tag{7}$$

We prove that, under suitable assumptions on the fields $\sigma_k(x)$ (those of Section 3.1), this stochastic point vortex dynamics is globally well posed (in particular coalescence of point vortices disappear) for *all* initial conditions. This is a stochastic improvement of the deterministic theory, as in [6].

For the improvements in well posedness of the linear transport equations considered in [6], it was sufficient to take constant fields σ_k. Here, to avoid point vortex coalescence, we need space-dependent fields with a high degree of hypoellipticity, a technically complex condition (see Hypothesis 1) which however is generically satisfied (see Section 4). Notice that in the trivial case when $\omega_j = 0$ for every $j = 1, ..., n$, system (7) is the so called n-point motion associated to the vector fields σ_k. A priori this is highly degenerate; for this reason we need suitable hypoellipticity conditions.

It would be trivial to improve the regularity of the deterministic system (5) by adding independent Brownian motions to each component, but this would not correspond to a Lagrangian point vortex formulation of stochastic Euler equations.

Let us mention that Kotelenez [10, Chapter 8] considered a similar stochastic perturbation of Euler equation and the associated point vortex dynamics with the aim to understand the physical interest and properties of the model. However he is not concerned with the regularizing properties of such kinds of noises.

Plan of the paper. In Section 2 we explain the formal relation between the stochastic Euler equation and the SDE for the point vortices; this relation fixes the form of the noise allowed in the SDE. In Section 3 we state our hypothesis and prove the main result about well-posedness of point vortex dynamics for all initial conditions. In Section 4 we give a self-contained proof of the fact that our hypothesis is generically satisfied. Last we gather in the appendix a series of well-known result on the density of the law of SDEs under various conditions on the vector fields.

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2 Stochastic 2D Euler equations and vortex dynamics

The aim of this section is to provide an heuristic motivation for system (7). A rigorous link between it and the original stochastic Euler equations is not given in this work. It is already a difficult problem in the deterministic case, where one of the best available justifications is the result which states that unique solutions of Euler equations corresponding to smoothing of the distributional initial vorticity, converge in the weak sense of measures to (4), see [15]. The result holds as far as point vortices do not coalesce.

Due to the difficulty of this subject, we do not aim here to give rigorous results on the link between (7) and (6), but only to provide an heuristic motivation. For this reason, the rest of this section is not always written in rigorous terms and we intentionally miss important details like functions spaces, regularity of functions, etc.

2.1 Deterministic case

In dimension 2, the vorticity field $\xi = \nabla^{\perp} \cdot u = \partial_2 u_1 - \partial_1 u_2$ satisfies equation (2) where $\xi_0 = \nabla^{\perp} \cdot u_0$. If φ (called potential) solves the equation

$$\Delta\varphi = -\xi$$

then $u = \nabla^{\perp}\varphi$ satisfies $\xi = \nabla^{\perp} \cdot u$. Hence, formally speaking, u can be reconstructed from ξ by the so called Biot-Savart law (3).

Depending on the fact that we consider the equations in full space with square integrable conditions at infinity, or on a torus with periodic boundary conditions or other cases, one can make rigorous and uniquely defined the previous procedure of reconstruction of u from ξ. Let us work on the 2D-torus $\mathbb{T} = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$. Denote by G the Green's function of $-\Delta$ on $\mathbb{T}$, then $G(x) = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \|k\|^{-2} e^{ik\cdot x}$. The distribution G is in fact a function, with a logarithmic divergence at $x = 0$, smooth everywhere else and satisfies (cf. page 18 of [15]):

$$\begin{gathered} C_1 \log|x| - C_3 \leq G(x) \leq C_2 \log|x| + C_3 \\ |DG(x)| \leq C_3|x|^{-1}, \quad |D^2G(x)| \leq C_3|x|^{-2} \end{gathered}$$

for all $x \in [-\pi, \pi]^2$, for some positive constants C_1, C_2, C_3.

Given a periodic field ξ with suitable regularity, a periodic (distributional) solution of $\Delta\varphi = -\xi$ is given by $\varphi(x) = \int_{\mathbb{T}} G(x-y)\,\xi(y)\,dy$. All other solutions differ by constants. The vector field $u = \nabla^{\perp}\varphi$ is thus uniquely defined from ξ:

$$u(x) = \int_{\mathbb{T}} K(x-y)\,\xi(y)\,dy \tag{8}$$

where

$$K(x) = \nabla^{\perp} G(x) = \sum_{k=(k_1,k_2)\in\mathbb{Z}^2\setminus\{0\}} \frac{ik^{\perp}}{\|k\|^2} e^{ik\cdot x}, \quad k^{\perp} = (k_2, -k_1).$$

Equation (8) is the integral form of Biot Savart law, used throughout the paper. Since u is the unique field such that $\xi = \nabla^{\perp}\cdot u$, it is the velocity field associated to ξ.

A limit case of vorticity field is the distributional one given by (4) where x_t^i is the position of point vortex i at time t and ω_i is its intensity (independent of time because vorticity is just transported). This distributional field does not satisfy Euler equations in the usual distributional sense: the nonlinear term (φ is a smooth test function)

$$\int \xi(x,t) u(x,t)\cdot\nabla\varphi(x) dx$$

is not well defined a priori, because the velocity field $u(x,t)$ associated to (4) is singular exactly at the delta Dirac points of ξ. However, there are limit arguments, see [15] which rigorously motivate the following closed set of equations for the positions of point vortices:

$$\frac{dx_t^i}{dt} = \sum_{j\neq i} \omega_j K(x_t^i - x_t^j), \quad i = 1, ..., n \tag{9}$$

Let us briefly explain this equation. Formally, point vortices are transported by the fluid, hence they should satisfy $\frac{dx_t^i}{dt} = u(x_t^i, t)$ where $u(x,t)$ is the velocity field associated to the vorticity field (4). If we put (4) in (8) we get

$$u(x,t) = \sum_{j=1}^{n} \omega_j K(x - x_t^j).$$

However, this expression is correct in all points x different from the vortex points themselves; notice that $K(x)$ diverges at $x = 0$ so $u(x,t)$ should be properly interpreted at $x = x_t^i$. It turns out (see [15]) that the correct interpretation is

$$u(x_t^i, t) = \sum_{j\neq i} \omega_j K(x_t^i - x_t^j). \tag{10}$$

Giving us eq. (9).

2.2 Lack of full well-posedness

Equations (9) are not trivial since the vector field is divergent when two particles collide; and there is no repulsion (but also no attraction) when particles approach each other. Nevertheless, in the periodic case, equations (9) are well-posed for almost every initial condition $X_0 = (x_0^1, ..., x_0^n)$ with respect to Lebesgue measure on the product space. One can interpret this result by saying that the system is almost surely well-posed when initial conditions are chosen at random in a uniform way.

In the whole space the same result is known under the additional assumption that $\{\omega_i\}_{i=1,...,n}$ satisfy $\sum_{i\in\pi} \omega_i \neq 0$ for all $\pi \subset \{1, ..., n\}$.

The restriction to almost every initial condition is not just a weakness of the technique: there are counterexamples in the form of explicit initial configurations which collide in finite time, see [15].

2.3 Stochastic case

We shall prove well-posedness for every initial condition under a suitable random perturbation. The idea behind is simply that the noise makes the same effect of a randomization of the initial conditions. However, there is a subtle difference between these two randomizations. In the case of initial conditions, Lebesgue measure in product space is used, thus the initial positions of particles are perturbed independently one of each other. In the noise case, it would be trivial to perturb each particle independently: this produces immediately a strong regularization which ultimately would imply well-posedness. Such kind of noise, however, has no meaning in terms of Euler equation. What we want is a noisy version of Euler equation which is solvable in the case of distributional point vortex fields. When we write the noise at the level of the point vortex dynamics, *the noise is the same for all vortices* (but since it is a space dependent noise, it is computed ad different spatial points). This is in principle a source of difficulties.

We consider the stochastic equation with multiplicative noise (6) where $\sigma_k(x)$ are 2D smooth vector fields and $\{\beta_t^k\}_{k=1,...,N}$ are a finite sequence of independent Brownian motions defined on a stochastic basis $(\Omega, F, (F_t), P)$ (fixed once and for all).

The associated dynamics of n point vortices is the stochastic system in $\mathbb{R}^{2n}$

$$dx_t^i = \sum_{j\neq i} \omega_j K(x_t^i - x_t^j)dt + \sum_{k=1}^{N} \sigma_k(x_t^i) \circ d\beta_t^k, \quad x_0^i = x^i \tag{11}$$

for $i = 1, ..., n$, with each single x_t^i in $\mathbb{R}^2$ and where $\circ$ denote Stratonovich integration.

Let us formally show that the measure valued vorticity field (4), with x_t^i given by a solution of equations (11), solve (6). The weak form of the stochastic Euler equation is

(assuming $\operatorname{div}\sigma_k = 0,\ k = 1, \ldots, N$)

$$d\langle \xi, \varphi \rangle = \langle \xi, u \cdot \nabla \varphi \rangle dt + \sum_{k=1}^{N} \langle \xi, \sigma_k \cdot \nabla \varphi \rangle \circ d\beta_t^k$$

where we have denoted by φ a smooth test function and by $\langle ., . \rangle$ the dual pairing. Let ξ be the distribution defined by (4). From Itô formula for $\varphi(x_t^i)$ in Stratonovich form we get

$$\begin{aligned} d\langle \xi, \varphi \rangle = \sum_{i=1}^{n} \omega_i d\varphi(x_t^i) &= \sum_{i=1}^{n} \omega_i \nabla \varphi(x_t^i) \sum_{j \neq i} \omega_j K(x_t^i - x_t^j) dt \\ &+ \sum_{i=1}^{n} \omega_i \sum_{k=1}^{N} \nabla \varphi(x_t^i) \sigma_k(x_t^i) \circ d\beta_t^k. \end{aligned}$$

The second term is the right one:

$$\sum_{k=1}^{N} \langle \xi, \sigma_k \cdot \nabla \varphi \rangle \circ d\beta_t^k = \sum_{i=1}^{n} \omega_i \sum_{k=1}^{N} \sigma_k(x_t^i) \cdot \nabla \varphi(x_t^i) \circ d\beta_t^k.$$

As to the first term, recall that the velocity field u associated to point vortices is given by (10). Then

$$\langle \xi, u \cdot \nabla \varphi \rangle = \sum_{i=1}^{n} \omega_i u(x_t^i) \cdot \nabla \varphi(x_t^i) = \sum_{i=1}^{n} \sum_{j \neq i} \omega_i \omega_j K(x_t^i - x_t^j) \cdot \nabla \varphi(x_t^i)$$

and thus also the first term is the right one. This completes the heuristic proof.

A very important remark, already mentioned in the previous section, is that the noise in this system is the same for all particles. Thus this is very similar to the so called n-point motion of a single SDE. The regularizing effect of the noise at the level of the n-point motion is a very non-trivial fact. For instance, the easiest non-degenerate noise, namely the simple additive one (σ_k are 2D vectors)

$$dx_t^i = \sum_{j \neq i} \omega_j K(x_t^i - x_t^j) dt + \sum_{k=1}^{N} \sigma_k d\beta_t^k$$

cannot yield any better result than the deterministic case, because the change of variables $y_t^i = x_t^i - \sum_{k=1}^{N} \sigma_k d\beta_t^k$ leads to the equation

$$dy_t^i = \sum_{j \neq i} \omega_j K(y_t^i - y_t^j) dt$$

which is exactly the deterministic one. If collapse happens for an initial condition of this equation, the same initial condition produces collapse in the previous SDE.

On the contrary, a strongly space dependent noise may contrast collapse. When point vortices come close one to the other, the noise should be sufficiently un-correlated (at small distances) to perturb in a generic way the motion of the two vortices and produce the same effect of a random perturbation of initial conditions.

3 Main results and proofs

3.1 Regularization by noise

We consider system (11) on the 2D-torus $\mathbb{T} = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$. It is a C^∞ compact connected Riemannian manifold with the smooth metric induced by Euclidean metric of $\mathbb{R}^2$. In fact, for simplicity, we may assume we work on the full space $\mathbb{R}^2$ and all the vector fields and functions are 2π-periodic, but sometimes the interpretation as a compact manifold is more illuminating. We will consider a fixed choice $\{\omega_j\}_{j=1,\ldots,n} \subset \mathbb{R}$ of vortex intensities.

Let Γ be the set of all $\left(x^1, ..., x^n\right) \in \mathbb{T}^n$ such that $x^i = x^j$ for some $i \neq j$ (Γ is the union of the generalized diagonals of $\mathbb{T}^n$). Let $\{\sigma_k\}_{k=1,\ldots,N}$ be a finite number of smooth vector fields on $\mathbb{T}$. Introduce the associated vector fields on $\mathbb{T}^n$:

$$A_{\sigma_k}(x^1, \ldots, x^n) = A_k(x^1, \ldots, x^n) = \left(\sigma_k(x^1), \ldots, \sigma_k(x^n)\right) \tag{12}$$

Recall that given vector fields A, B in $\mathbb{R}^m$, their Lie bracket $[A, B]$ is the vector fields in $\mathbb{R}^m$ defined by

$$[A, B] = (A \cdot \nabla)B - (B \cdot \nabla)A.$$

We assume that $\{\sigma_k\}_{k=1,\ldots,N}$ satisfies:

Hypothesis 1

1. *The vector fields σ_k are periodic, infinitely differentiable and* $\mathrm{div}\sigma_k = 0$ *;*
2. *(Bracket generating condition) The vector space spanned by the vector fields*

$$A_1, ..., A_N, \qquad [A_i, A_j], 1 \leq i, j \leq N, \qquad [A_i, [A_j, A_k]], 1 \leq i, j, k \leq N, ...$$

at every point $x \in \Gamma^c$ is $\mathbb{R}^{2n}$.

The second assumption is a form of Hörmander's condition. It will ensures that the law at any time $t > 0$ of the solution of a regularized stochastic equation is absolutely continuous with respect to Lebesgue measure if we start outside the diagonal Γ (see also the appendix).

Under this hypothesis we are able to prove the following result of well-posedness of the dynamics for all initial n-point configurations.

Theorem 1 *Under Hypothesis 1, for all $X_0 = (x_0^1, ..., x_0^n) \in \mathbb{T}^n \backslash \Gamma$ equation (11) has one and only one global strong solution.*

Before going to the proofs, let us make some remarks on our hypothesis. The bracket generation condition appears already in the papers [2] and [5] which study the asymptotic behavior of stochastic flows (among other properties, the Lyapunov exponents and the large deviations for additive functionals). In [5, Section 2] it is stated that the bracket generating condition in Hypothesis 1 is *generic* among smooth vector fields. We have been unable to find a proof of this statement in the literature (both stochastic or more dynamical-system oriented) and so in Section 4 we give a self-contained and elementary argument which justifies this statement.

Remark 2 *Instead of the noise taken from [2],[5], it is natural to consider an infinite dimensional noise $W(x,t) = \sum_{k=1}^{\infty} \sigma_k(x)\beta_t^k$, for instance the isotropic divergence free Brownian field which generates the isotropic Brownian motion, see [1],[11],[12]. Here we restrict our attention to finite dimensional noise for which we already know results about absolute continuity of fixed time marginals. The interesting fact about infinite dimensional noise is that it is easy to constructs explicit noises which are "full" outside Γ and for which it is reasonable to expect the validity of density results on the law.*

The proof of Theorem 1 goes through the study of a regularized problem where the singular Biot-Savart kernel is replaced by a smooth one. We first prove well-posedness for almost every initial conditions (as in the deterministic setting) and then, exploiting the existence of a density for the law at fixed time, we improve to well-posedness for all initial conditions.

3.2 Regularization

For sufficiently small δ, let $G^\delta(x)$ be a smooth 2π-periodic function (hence bounded with its derivatives) such that, on $[-\pi,\pi]^2$,

$$G^\delta(x) = G(x) \text{ for } |x| > \delta.$$

Set $K^\delta = \nabla^\perp G^\delta$. We shall use the following quantitative properties, beside smoothness:

$$C_1 \log(|x| \vee \delta) - C_3 \leq G^\delta(x) \leq C_2 \log(|x| \vee \delta) + C_3$$
$$|DG^\delta(x)| \leq C_3(|x| \vee \delta)^{-1}, \quad |D^2 G^\delta(x)| \leq C_3(|x| \vee \delta)^{-2}$$

for all $x \in [-\pi,\pi]^2$, for some positive constants C_1, C_2, C_3 (possibly different from those of the same inequalities for G but independent of δ). We consider the regularized equation

$$dx_t^{i,\delta} = \sum_{j\neq i} \omega_j K^\delta(x_t^{i,\delta} - x_t^{j,\delta})dt + \sum_{k=1}^{N} \sigma_k(x_t^{i,\delta}) \circ d\beta_t^k. \tag{13}$$

which in Itô form reads

$$dx_t^{i,\delta} = \sum_{j\neq i} \omega_j K^\delta(x_t^{i,\delta} - x_t^{j,\delta})dt + \frac{1}{2}\sum_{k=1}^{N}(\sigma_k \cdot \nabla\sigma_k)(x_t^{i,\delta})dt + \sum_{k=1}^{N}\sigma_k(x_t^{i,\delta})d\beta_t^k. \tag{14}$$

We immediately have: for every $X_0 = (x_0^1, ..., x_0^n) \in \mathbb{R}^{2n}$, there exists a unique strong solution $(X_t^{X_0})_{t\geq 0}$ to this equation in $\mathbb{R}^{2n}$. We even have a smooth stochastic flow φ_t^δ on $\mathbb{T}^n$, see [9].

3.3 Measure conservation

Denote the divergence in $\mathbb{R}^2$ by div_2 and in $\mathbb{R}^{2n}$ by div_{2n}.We have

$$\mathrm{div}_{2n}[\sum_{j\neq i}\omega^i K^\delta(x_i - x_j)]_{i=1,\ldots,n} = \sum_{i=1}^{n}\mathrm{div}_2[\sum_{j\neq i}\omega^i K^\delta(x_i - x_j)] = 0$$

because $K^\delta = \nabla^\perp G^\delta$. The same is true without regularization. Moreover,

$$\mathrm{div}_{2n}[\sum_{k=1}^{N}\sigma_k(x^i)\beta_t^k]_{i=1,\ldots,n} = \sum_{i=1}^{n}\mathrm{div}_2\sigma_k(x^i)\beta_t^k = 0$$

because $\mathrm{div}_2\sigma_k = 0$. By classical computations on the smooth flow φ_t^δ, one can check that its Jacobian determinant is equal to one, as a consequence of the previous divergence free conditions. Hence we have:

Lemma 3 *For every integrable function h on $\mathbb{T}^n$, we have*

$$\int_{\mathbb{T}^n} h(\varphi_t^\delta(X_0))dX_0 = \int_{\mathbb{T}^n} h(Y)dY.$$

3.4 Estimates about coalescence

Denote by $[x, y]_t$ the mutual quadratic covariation of two continuous semimartingales $(x_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$. Denote by x^α, $\alpha = 1, 2$, the two components of an element of $\mathbb{T}$ in the coordinate frame coming from Euclidean coordinates. If $(x_t^{i,\delta})$ is a solution of equation (13), then

$$[(x^{i,\delta} - x^{j,\delta})^\alpha, (x^{i,\delta} - x^{j,\delta})^\beta]_t = \sum_{k=1}^{N}\int_0^t (\sigma_k^\alpha(x_s^{i,\delta}) - \sigma_k^\alpha(x_s^{j,\delta}))(\sigma_k^\beta(x_s^{i,\delta}) - \sigma_k^\beta(x_s^{j,\delta}))ds.$$

Lemma 4 *Let φ_t^δ, be the flow on $\mathbb{T}^n$ associated to (13). Let $g^\delta : \mathbb{T}^n \to \mathbb{R}$ be the function*

$$g^\delta(X) = - \sum_{i,j=1...,n,\ i\neq j} G^\delta(x^i - x^j), \quad X = (x^1, ..., x^n).$$

Then there exists a non negative integrable function h on $\mathbb{T}^n$ such that

$$E[\sup_{t\in[0,T]} g^\delta(\varphi_t^\delta(X_0))] \leq g^\delta(X_0) + \int_0^T E[h(\varphi_t^\delta(X_0))]dt.$$

Proof. From Itô formula we have

$$g^\delta(\varphi_t^\delta(X_0)) = g^\delta(X_0) - \sum_{i,j=1...,n,\ i\neq j} (I_{1a}^{ij}(t) + I_{1b}^{ij}(t) + I_2^{ij}(t) + I_3^{ij}(t) + I_4^{ij}(t))$$

where

$$I_{1a}^{ij}(t) = \int_0^t \sum_{i'\neq i} \omega_{i'} K^\delta(x_s^{i,\delta} - x_s^{i',\delta}) \cdot \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta})ds$$

$$I_{1b}^{ij}(t) = -\int_0^t \sum_{j'\neq j} \omega_{j'} K^\delta(x_s^{j,\delta} - x_s^{j',\delta}) \cdot \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta})ds$$

$$I_2^{ij}(t) = \sum_{k=1}^N \int_0^t (\sigma_k(x_s^{i,\delta}) - \sigma_k(x_s^{j,\delta})) \cdot \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta})d\beta_s^k$$

$$I_3^{ij}(t) = \frac{1}{2} \sum_{\alpha,\beta=1}^2 \int_0^t \frac{\partial^2 G^\delta}{\partial x^\alpha \partial x^\beta}(x_s^{i,\delta} - x_s^{j,\delta})d[(x^{i,\delta} - x^{j,\delta})^\alpha, (x^{i,\delta} - x^{j,\delta})^\beta]_s$$

$$I_4^{ij}(t) = \frac{1}{2} \int_0^t \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta}) \cdot [(\sigma_k \cdot \nabla\sigma_k)(x_s^{i,\delta}) - (\sigma_k \cdot \nabla\sigma_k)(x_s^{j,\delta})]ds.$$

Since $K^\delta = \nabla^\perp G^\delta$ and $\nabla^\perp G^\delta(x)$ is orthogonal to $\nabla G^\delta(x)$, we have

$$I_{1a}^{ij}(t) = \int_0^t \sum_{i'\neq i,\ i'\neq j} \omega_{i'} K^\delta(x_s^{i,\delta} - x_s^{i',\delta}) \cdot \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta})ds$$

$$I_{1b}^{ij}(t) = -\int_0^t \sum_{j'\neq j,\ j'\neq i} \omega_{j'} K^\delta(x_s^{j,\delta} - x_s^{j',\delta}) \cdot \nabla G^\delta(x_s^{i,\delta} - x_s^{j,\delta})ds.$$

Hence

$$|I_{1a}^{ij}(t)| \leq C \int_0^t \sum_{i'\neq i,\ i'\neq j} (|x_s^{i,\delta} - x_s^{i',\delta}| \vee \delta)^{-1} (|x_s^{i,\delta} - x_s^{j,\delta}| \vee \delta)^{-1} ds$$

$$|I_{1b}^{ij}(t)| \leq C \int_0^t \sum_{j' \neq j,\, j' \neq i} (|x_s^{j,\delta} - x_s^{j',\delta}| \vee \delta)^{-1} (|x_s^{i,\delta} - x_s^{j,\delta}| \vee \delta)^{-1} ds$$

$$\sum_{i,j=1\ldots,n,\, i\neq j} (|I_{1a}^{ij}(t)| + |I_{1b}^{ij}(t)|) \leq C \int_0^t h_1^\delta(\varphi_s^\delta(X_0)) ds$$

where

$$h_1^\delta(X) = \sum_{\substack{i,j,l=1\ldots,n \\ i\neq j,\, l\neq i,\, l\neq j}} (|x^i - x^l| \vee \delta)^{-1} (|x^i - x^j| \vee \delta)^{-1}$$

with $X = (x^1, ..., x^n)$. Setting

$$h_1(X) = \sum_{\substack{i,j,l=1\ldots,n \\ i\neq j,\, l\neq i,\, l\neq j}} (|x^i - x^l|)^{-1} (|x^i - x^j|)^{-1}$$

we have that h_1 is integrable over $\mathbb{T}^n$, and $h_1^\delta(X) \leq h_1(X)$ for all $X \in \mathbb{T}^n$. Moreover, By BDG inequality and the smoothness of σ_k we have

$$E\left[\sup_{t\in[0,T]} \left| I_2^{ij}(t) \right| \right] \leq CE\left[\left[I_2^{ij}, I_2^{ij} \right]_T^{1/2} \right]$$

$$\leq CE\Big[\Big(\int_0^T \left(\left| x_s^{i,\delta} - x_s^{j,\delta} \right| \vee \delta \right)^{-2} \left| x_s^{i,\delta} - x_s^{j,\delta} \right|^2 ds \Big)^{1/2} \Big]$$

hence $\sum_{i,j=1\ldots,n,\, i\neq j} E[\sup_{t\in[0,T]} |I_2^{ij}(t)|] \leq C$. Finally,

$$\sup_{t\in[0,T]} |I_3^{ij}(t)| \leq C \int_0^T (|x_s^{i,\delta} - x_s^{j,\delta}| \vee \delta)^{-2} |x_s^{i,\delta} - x_s^{j,\delta}|^2 ds$$

and

$$\sup_{t\in[0,T]} |I_4^{ij}(t)| \leq C \int_0^T (|x_s^{i,\delta} - x_s^{j,\delta}| \vee \delta)^{-1} |x_s^{i,\delta} - x_s^{j,\delta}| ds$$

so again $\sum_{i,j=1\ldots,n,\, i\neq j} (|I_3^{ij}(t)| + |I_4^{ij}(t)|) \leq C$. Summarizing,

$$E\left[\sup_{t\in[0,T]} g^\delta\left(\varphi_t^\delta(X_0) \right) \right] \leq g^\delta(X_0)$$

$$+ \sum_{i,j=1\ldots,n,\, i\neq j} E[\sup_{t\in[0,T]} (|I_{1a}^{ij}(t)| + |I_{1b}^{ij}(t)| + |I_2^{ij}(t)| + |I_3^{ij}(t)| + |I_4^{ij}(t)|)]$$

$$\leq g^\delta(X_0) + C \int_0^T E\left[h_1\left(\varphi_s^\delta(X_0) \right) \right] ds + C.$$

The proof is complete. $\blacksquare$

Corollary 5 *There is a constant $C > 0$ such that for every $\delta > 0$*

$$E \int_{\mathbb{T}^n} \sup_{t \in [0,T]} [g^{\delta}(\varphi_t^{\delta}(X_0))] dX_0 \leq C < \infty.$$

Proof. From the previous lemma and lemma 3 we have

$$\begin{aligned} E \int_{\mathbb{T}^n} \sup_{t \in [0,T]} [g^{\delta}(\varphi_t^{\delta}(X_0))] dX_0 &\leq \int_{\mathbb{T}^n} g^{\delta}(X_0) dX_0 + E \int_0^T [\int_{\mathbb{T}^n} h(\varphi_t^{\delta}(X_0)) dX_0] dt \\ &= \int_{\mathbb{T}^n} g^{\delta}(X_0) dX_0 + T \int_{\mathbb{T}^n} h(Y) dY. \end{aligned}$$

The proof is complete. ■

Corollary 6 *There exists a constant $C > 0$ such that for all $\varepsilon, \delta > 0$ we have*

$$(\lambda_{\mathbb{T}^n} \otimes P)(\inf_{i \neq j} \inf_{t \in [0,T]} |x_t^{i,\delta} - x_t^{j,\delta}| \leq \varepsilon) \leq -\frac{C}{\log(\varepsilon \vee \delta)}.$$

Proof. We have

$$\begin{aligned} g^{\delta}(\varphi_t^{\delta}(X_0)) &= -\sum_{i,j=1\ldots,n,\, i \neq j} G^{\delta}(x_t^{i,\delta} - x_t^{j,\delta}) \\ &\geq -\sum_{i,j=1\ldots,n,\, i \neq j} C_2 \log(|x_t^{i,\delta} - x_t^{j,\delta}| \vee \delta) - \frac{n(n-1)}{2} C_3. \end{aligned}$$

Given $\varepsilon, \delta > 0$, smaller than one, if $\inf_{i \neq j} \inf_{t \in [0,T]} |x_t^{i,\delta} - x_t^{j,\delta}| \leq \varepsilon$ namely if there are $t_0 \in [0,T]$ and $i_0 \neq j_0$ such that $|x_{t_0}^{i_0,\delta} - x_{t_0}^{j_0,\delta}| \leq \varepsilon$ then

$$g^{\delta}(\varphi_{t_0}^{\delta}(X_0)) \geq -C_2 \log(\varepsilon \vee \delta) - \frac{n(n-1)}{2}(C_2 \log 2\pi\sqrt{2} + C_3)$$

(we have used the fact that $\log(|x_t^{i,\delta} - x_t^{j,\delta}| \vee \delta) \leq \log 2\pi\sqrt{2})$ and thus

$$\sup_{t \in [0,T]} g^{\delta}(\varphi_t^{\delta}(X_0)) \geq -C_2 \log(\varepsilon \vee \delta) - C_4 n^2.$$

By Chebyshev inequality (notice that $-C_2 \log(\varepsilon \vee \delta) > 0$) and the previous lemma,

$$\begin{aligned} &(\lambda_{\mathbb{T}^n} \otimes P)(\inf_{i \neq j} \inf_{t \in [0,T]} |x_t^{i,\delta} - x_t^{j,\delta}| \leq \varepsilon) \\ &\leq (\lambda_{\mathbb{T}^n} \otimes P)(\sup_{t \in [0,T]} g^{\delta}(\varphi_t^{\delta}(X_0)) + C_4 n^2 \geq -C_2 \log(\varepsilon \vee \delta)) \\ &\leq -\frac{C_5 n^2}{\log(\varepsilon \vee \delta)}. \end{aligned}$$

The proof is complete. ■

Remark 7 *The function h and the constants C of the previous statements depend on the number n of point vortices and the time interval $[0,T]$.*

3.5 Well-posedness for Lebesgue almost every initial condition

As a first consequence of the previous estimates, we prove the same result of the deterministic case.

Recall that Γ is the singular set in $\mathbb{T}^n$ for the vortex dynamics, namely the set of all $(x^1,...,x^n) \in \mathbb{T}^n$ such that $x^i = x^j$ for some $i \neq j$. The drift of equation (11) is well defined only on Γ^c. Thus the notion of strong solution $(X_t)_{t\geq 0}$ to equation (11) is the classical one for SDEs with the addition of the condition that

$$P(X_t \in \Gamma^c \text{ for all } t \geq 0) = 1.$$

Theorem 8 *For Lebesgue almost every $X_0 = (x_0^1,...,x_0^n) \in \mathbb{T}^n$, equation (11) has one and only one global strong solution.*

Proof. Denote by Γ_δ the closed δ-neighbor of Γ in $\mathbb{T}^n$. Given $X_0 \in \Gamma_\delta^c$, denote by $\tau_{X_0}^\delta(\omega)$ the first instant when $\varphi_t^\delta(X_0) \in \Gamma_\delta$ and set it equal to $+\infty$ if this fact never happens. We have $P\left(\tau_{X_0}^\delta > 0\right) = 1$ by continuity of trajectories. The solution $\varphi_t^\delta(X_0)$, on the random interval $\left[0, \tau_{X_0}^\delta\right]$, is also the unique solution (X_t) of equation (11). Thus $\tau_{X_0}^\delta(\omega)$ is also the first instant when $X_t \in \Gamma_\delta$. Set

$$\tau_{X_0}(\omega) = \sup_{\delta\in(0,1)} \tau_{X_0}^\delta(\omega).$$

By localization, we have a unique solution of equation (11) on $[0, \tau_{X_0})$. If we prove that $P(\tau_{X_0} = \infty) = 1$ for a.e. X_0, we have proved the theorem. Given $T > 0$ and $\delta^* > 0$, it is sufficient to prove that for Lebesgue a.e. $X_0 \in \Gamma_{\delta^*}^c$, we have $P(\tau_{X_0} \geq T) = 1$.

Form the last corollary of the previous section we know that

$$(\lambda_{\mathbb{T}^n} \otimes P)(\inf_{i\neq j} \inf_{t\in[0,T]} |x_t^{i,\delta} - x_t^{j,\delta}| \leq \delta) \leq -\frac{C}{\log \delta}.$$

Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence such that the series $\sum_{k=1}^{\infty} \frac{1}{\log \delta_k}$ converges. Take it such that $\delta_k \leq \delta^*$ for all $k \in \mathbb{N}$. By Borel-Cantelli lemma, there is a measurable set $N \subset \mathbb{T}^n \times \Omega$ with $(\lambda_{\mathbb{T}^n} \otimes P)(N) = 0$, such that for all $(X_0, \omega) \in N^c$ there is $k_0 = k_0(X_0, \omega) \in \mathbb{N}$ such that for all $k \geq k_0(X_0, \omega)$

$$\inf_{i\neq j} \inf_{t\in[0,T]} |\varphi_t^{i,\delta_k}(X_0)(\omega) - \varphi_t^{j,\delta_k}(X_0)(\omega)| > \delta_k$$

where $\varphi_t^{i,\delta_k}(X_0)$ is x_t^{i,δ_k} when the initial condition is X_0. If we restrict ourselves to $(X_0, \omega) \in N^c \cap (\Gamma_{\delta^*}^c \times \Omega)$, the previous statement implies $\tau_{X_0}^{\delta_k}(\omega) \geq T$ for all $k \geq k_0(X_0, \omega)$. This implies

$$\tau_{X_0}(\omega) \geq T.$$

We have proved this inequality for all $(X_0, \omega) \in N^c \cap (\Gamma_{\delta^*}^c \times \Omega)$, namely for almost every (X_0, ω) in $\Gamma_{\delta^*}^c \times \Omega$. By Fubini-Tonelli theorem, there is a measurable set $\Delta \subset \Gamma_{\delta^*}^c$ with $\lambda_{\mathbb{T}^n}(\Delta) = 1$, such that for all $X_0 \in \Delta$ we have $\tau_{X_0}(\omega) \geq T$ with P-probability one. The proof is complete. ■

3.6 Improvement due to the noise

We may now prove our main result, Theorem 1.

Proof. (Theorem 1) Given $X_0 \in \Gamma^c$, a strong unique local solution on $[0, \tau_{X_0})$ exists (τ_{X_0} defined in the proof of theorem 8). Let us add a point Δ to $\mathbb{T}^n$ and set $\varphi_t(X_0) = \Delta$ for $t \geq \tau_{X_0}$, where $\tau_{X_0} < \infty$. The family of processes $\varphi_t(X_0)$, $X_0 \in \Gamma^c$, so defined, lives in $\Gamma^c \cup \Delta$ for positive times and is Markov. Then

$$P(\varphi_{[\varepsilon,T]}(X_0) \in \Gamma^c) = \int_{\Gamma^c \cup \{\Delta\}} P(\varphi_{[0,T-\varepsilon]}(Y) \in \Gamma^c) \mu_{\varphi_\varepsilon(X_0)}(dY)$$

where $\{\varphi_{[\varepsilon,T]}(X_0) \in \Gamma^c\} = \{\omega \in \Omega \ : \ \varphi_t(X_0)(\omega) \in \Gamma^c, \text{ for any } \ t \in [\varepsilon, T]\}$ and $\mu_{\varphi_\varepsilon(X_0)}$ is the law of $\varphi_\varepsilon(X_0)$. Denote by $N \subset \mathbb{T}^n$ a measurable set such that all initial conditions in N^c give rise to a well posed Cauchy problem. We have

$$P(\varphi_{[0,T-\varepsilon]}(Y) \in \Gamma^c) = 1$$

for all $Y \in N^c$. Then

$$\begin{aligned} P(\varphi_{[\varepsilon,T]}(X_0) \in \Gamma^c) &\geq \int_{N^c} P(\varphi_{[0,T-\varepsilon]}(Y) \in \Gamma^c) \mu_{\varphi_\varepsilon(X_0)}(dY) \\ &= 1 - \mu_{\varphi_\varepsilon(X_0)}(N). \end{aligned}$$

Now, assume $X_0 \in \Gamma^c_{\delta^*}$ for some $\delta^* > 0$. We have, for all $\delta \in (0, \delta^*)$,

$$\begin{aligned} \mu_{\varphi_\varepsilon(X_0)}(N) &= P(\varphi_\varepsilon(X_0) \in N) \\ &= P(\varphi_\varepsilon(X_0) \in N, \tau^\delta_{X_0} > \varepsilon) + P(\varphi_\varepsilon(X_0) \in N, \tau^\delta_{X_0} \leq \varepsilon) \\ &\leq P(\varphi^\delta_\varepsilon(X_0) \in N, \tau^\delta_{X_0} > \varepsilon) + P(\tau^\delta_{X_0} \leq \varepsilon) \\ &\leq P(\varphi^\delta_\varepsilon(X_0) \in N) + P(\tau^\delta_{X_0} \leq \varepsilon) \\ &= P(\tau^\delta_{X_0} \leq \varepsilon). \end{aligned}$$

To say that $P(\varphi^\delta_\varepsilon(X_0) \in N) = 0$ we have used two facts: N is Lebesgue-negligible, the law of $\varphi^\delta_t(X_0)$ on $\mathbb{T}^n$ is absolutely continuous with respect to Lebesgue measure, for each $X_0 \in \Gamma^c$, $\delta > 0$, $t > 0$. The latter property is a consequence of the second main assumption of section 3.1. See the appendix A for details; we apply, in particular, Theorem 15.

Just by continuity of trajectories, we have $\lim_{\varepsilon \to 0} P(\tau^\delta_{X_0} \leq \varepsilon) = 0$. Hence

$$\lim_{\varepsilon \to 0} P(\varphi_{[\varepsilon,T]}(X_0) \in \Gamma^c) = 1.$$

The family of events $(\varphi_{[\frac{1}{n},T]}(X_0) \in \Gamma^c)$ is decreasing in n, hence $P(\varphi_{[\frac{1}{n},T]}(X_0) \in \Gamma^c)$ is also decreasing. This implies $P(\varphi_{[\varepsilon,T]}(X_0) \in \Gamma^c) = 1$ for every ε giving $P(\varphi_{[0,T]}(X_0) \in \Gamma^c) = 1$. ∎

3.7 Variations on the result

Let us complete this section with a variant of the previous result. Next section is devoted to the proof that the assumptions of Section 3.1 are generic. But in fact we prove more, namely that generically it happens that the vector fields $A_1, ..., A_N$ themselves span $\mathbb{R}^{2n}$ at every point $x \in \Gamma^c$ is $\mathbb{R}^{2n}$ (no Lie brackets are needed). It is thus meaningful to investigate the problem under the following assumption: $\{\sigma_k\}_{k=1,...,N}$ satisfies:

Hypothesis 2

1. *σ_k are periodic, C^2 and* $\mathrm{div}\sigma_k = 0$
2. *the vector space spanned by the vector fields $A_1, ..., A_N$ at every point $x \in \Gamma^c$ is $\mathbb{R}^{2n}$.*

Item 2 of this assumption is more restrictive than the corresponding one of Hypothesis 1, but smoothness of the fields is no more needed. Under Hypothesis 2, we still have that the law of $\varphi_t^\delta(X_0)$ on $\mathbb{T}^n$ (see the notations of the previous sections) is absolutely continuous with respect to Lebesgue measure, for each $X_0 \in \Gamma^c$, $\delta > 0$, $t > 0$; we use now Corollary 18. Let us also remark that the proof of absolute continuity of the law under this assumption is more elementary than under Hypothesis 1. For all these reasons it is worth to state also the following variant of Theorem 1 (the proof is the same, based on the previous remark on the absolute continuity).

Theorem 9 *Under Hypothesis 2, for all $X_0 = (x_0^1, ..., x_0^n) \in \mathbb{T}^n \backslash \Gamma$ equation (11) has one and only one global strong solution.*

4 Generic n-point motions are hypoelliptic

In this section we are going to provide a self-contained proof of the following statement which stipulates that n-point motions satisfying our assumptions are *generic*.

Theorem 10 *For all $M > 2n$ there exists a residual set $Q \subset (C^\infty)^{2nM}$ such that for every $(f_{a,i})_{a=1,...,M,i=1,...,2n} \in Q$ we have $\mathrm{span}\{A_{f_{a,i}}(x)\}_{a=1,...,M,i=1,...,2n} = \mathbb{R}^{2n}$ for every $x \in \Gamma^c$.*

The parametric Sard's theorem (or Thom's transversality theorem) are general tools which allow to prove generic properties of geometric objects (see, for instance [7]); here we intend properties valid for almost all objects with respect to some natural measure, or valid in a residual set (countable intersection of open dense sets). For some applications of transversality to control theory the reader could look at [8] where some interesting examples are worked out in a quite explicit setting.

Here we consider an easy version of the theorem which we are going to use to show that for a sufficiently large but otherwise generic family of vector fields on the torus, the

associated n-point motion generate, as a Lie algebra, the full tangent space in each point outside the diagonals. The basic idea is simple, unfortunately we haven't found a reference to an equivalent statement which do not require some background in differential topology to be understood, so we provide here the easy proof for reader sake.

Theorem 11 *Let $\ell < n$ and let $X \subset \mathbb{R}^\ell$ and $Y \subset \mathbb{R}^m$ be open sets. Consider a C^1 function $F : X \times Y \to \mathbb{R}^n$ and assume that 0 is a* regular value *for F (i.e. the Jacobian matrix $DF(x,y)$ is surjective for all $(x,y) \in F^{-1}(0)$). Then the set $\mathcal{X}_y = \{x \in X : F(x,y) = 0\}$ is empty for Lebesgue almost every $y \in Y$.*

Proof. Consider a point $(x_0, y_0) \in F^{-1}(0)$. By the implicit function theorem and the fact that $\dim \operatorname{Im} DF(x_0, y_0) = n$ there exists open sets U, V such that $x_0 \in U \subset X$ and $y_0 \in V \subset Y$ and for which the set $(U \times V) \cap F^{-1}(0)$ is the graph of a C^1 function defined on the open set $W \subset \mathbb{R}^{\ell+m-n}$. In particular there exists an differentiable homeomorphism $\psi : W \to (U \times V) \cap F^{-1}(0) \subset \mathbb{R}^\ell \times \mathbb{R}^m$. Let $\pi_2 : X \times Y \to Y$ be the canonical projection over the second factor and consider the differentiable map $\pi_1 \circ \psi : W \to \mathbb{R}^m$: the image $W' = \pi_1 \circ \psi(W) \subset V$ of W is a set of dimension $m + \ell - n < m$ and then of zero measure with respect to the m-dimensional Lebesgue measure. From an covering of $F^{-1}(0)$ by open sets of the form $U \times V$ we can then obtain a finite subcover and form the union of the associated W's which we call $\tilde{Y} \subset \mathbb{R}^m$ and which is still a negligible set. Now $\tilde{Y}$ contains exactly the points $y \in Y$ such that there exists $x \in X$ for which $F(x,y) = 0$, so we conclude that $y \in Y \backslash \tilde{Y} \Rightarrow \mathcal{X}_y = \emptyset$. ■

For every $d \in \mathbb{N}$ define the finite-dimensional real vector space $\mathcal{F}_d$ of the solenoidal vector fields $f : \mathbb{T} \to \mathbb{R}^2$ on the torus $\mathbb{T}$ of the form $f(x) = \sum_{|k_1|,|k_2| \leqslant d} k^\perp e^{i\langle k,x\rangle} \hat{f}(k)$. Fix $n \geqslant 1$ and recall that $\Gamma = \{(x^1, \ldots, x^n) : \min_{i \neq j} |x^i - x^j| = 0\} \subseteq \mathbb{T}^n$. Let $D = \dim(\mathcal{F}_d) = (2d+1)^2$. According to (12), for every vector field $f : \mathbb{T} \to \mathbb{R}^2$ on $\mathbb{T}$ define A_f as the vector field on $\mathbb{T}^n$ given by $A_f(x) = (f(x^1), \ldots, f(x^n))$.

To understand how to use Theorem 11 to prove genericity results for vector fields let us give a simple result which helps in understanding the main argument.

Lemma 12 *Let $d \in \mathbb{N}$. Fix a point $x \in \mathbb{T}^{2n}$ and assume that there exist vector fields $h_1, \ldots, h_{2n} \in \mathcal{F}_d$ such that the family $\{A_{h_i}(x)\}_{i=1,\ldots,2n}$ span all $\mathbb{R}^{2n}$. Then the same is true for Lebesgue almost every vector fields $\sigma_1, \ldots, \sigma_{2n} \in \mathcal{F}_d$ (i.e., we have that $\{A_{\sigma_i}(x)\}_{i=1,\ldots,2n}$ spans all $\mathbb{R}^2$, for a.e. $\sigma_1, \ldots, \sigma_{2n} \in \mathcal{F}_d$).*

Proof. Consider the map $\Psi : \mathcal{F}_d^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times \mathbb{R}$

$$\Psi(\sigma_1, \ldots, \sigma_{2n}, u) = (u_1 A_{\sigma_1}(x) + \cdots + u_{2n} A_{\sigma_{2n}}(x), \sum_{i=1}^{2n} u_i^2 - 1).$$

If we show that $\operatorname{rank}(D\Psi(\sigma_1, \ldots, \sigma_{2n}, u)) = 2n + 1$ for every $(\sigma_1, \ldots, \sigma_{2n}, u) \in \Psi^{-1}(0,0)$ then we have that the set of vector fields $\sigma_1, \ldots, \sigma_{2n}$ such that $\Psi(\sigma_1, \ldots, \sigma_{2n}, u) = 0$ for

some $u \in \mathbb{R}^{2n}$ such that $|u| = 1$ is of zero Lebesgue measure which allows us to conclude that $\{A_{\sigma_i}(x)\}_{i=1,\dots,2n}$ span all $\mathbb{R}^{2n}$ for almost every choice σ in $\mathcal{F}_d^{2n}$. Let us then compute $D\Psi$. Taking $q = (q_1, q_2) \in \mathbb{Z}^2$, $|q_1| \le d$, $|q_2| \le d$, we have

$$D_{\widehat{\sigma}_i(q)}\Psi(\sigma_1, \dots, \sigma_{2n}, u) = (u_i D_{\widehat{\sigma}_i(q)} A_{\sigma_i}(x), 0) = (u_i(q^\perp e^{i\langle q, x^1\rangle}, \dots, q^\perp e^{i\langle q, x^n\rangle}), 0)$$

and

$$D_{u_i}\Psi(\sigma_1, \dots, \sigma_{2n}, u) = (A_{\sigma_i}(x), 2u_i)$$

Since $|u|^2 = 1$ at least one of the components $u_i \neq 0$ so that the span of these vectors contains all the elements of the form $(h(x^1), \dots, h(x^n)) = (A_h(x), \rho)$ for arbitrary $h \in \mathcal{F}_d$ and $\rho \in \mathbb{R}$. But by assumption these vectors span $\mathbb{R}^{2n} \times \mathbb{R}$ so we can conclude using Theorem 11. ■

Let us now return to our main aim: build families of vector fields spanning $\mathbb{R}^{2n}$ in each point of Γ^c. The neighborhoods of the diagonals Γ are source of troubles so for the moment let us restrict to the consideration of n-point configurations belonging to an open set $G \subset \mathbb{T}^{2n}$ away from them.

Consider the map $\Phi : \mathcal{F}_d^{2nM} \times G \times (\mathbb{R}^{2n})^M \to (\mathbb{R}^{2n} \times \mathbb{R})^M$ given by

$$\Phi(F, x, U) = ((\sum_{i=1}^{2n} u_{1,i} A_{f_{1,i}}(x), |u_1|^2 - 1), \cdots, (\sum_{i=1}^{2n} u_{M,i} A_{f_{M,i}}(x), |u_M|^2 - 1)).$$

where $F = (f_{1,1}, \dots f_{1,2n}, \dots, f_{M,2n}) \in \mathcal{F}_d^{2nM}$ and $U = (u_1, \dots, u_M) \in (\mathbb{R}_0^{2n})^M$. Then

$$D\Phi(F, x, U) : \mathbb{R}^{2nMD} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nM} \to (\mathbb{R}^{2n} \times \mathbb{R})^M$$

The various components of the Jacobian matrix are given by (we denote by $\mathbb{I}_{a=b}$ the indicator function)

$$\begin{aligned}
(D_{u_{a,i}}\Phi(F, x, U))_b &= \mathbb{I}_{a=b}(A_{f_{a,i}}(x), 2u_{a,i}), \\
(D_{\hat{f}_{a,i}(q)}\Phi(F, x, U))_b &= \mathbb{I}_{a=b}(u_{a,i} D_{\hat{f}_{a,i}(q)}(f_{a,i}(x^1), \dots, f_{a,i}(x^n)), 0) \\
&= \mathbb{I}_{a=b}(u_{a,i}(q^\perp e^{i\langle q, x^1\rangle}, \dots, q^\perp e^{i\langle q, x^n\rangle}), 0) \\
(D_{x^i}\Phi(F, x, U))_a &= \sum_{j=1}^{2n} (u_{a,j}(f_{a,j}(x^1), \dots, D_{x^i} f_{a,j}(x^i), \dots, f_{a,j}(x^n)), 0)
\end{aligned}$$

where $a, b = 1, \dots, M$. The image of $D\Phi(F, x, U)$ contains then vectors v of the form

$$v_b = \sum_{a,i,q} \lambda_{a,i,q} (D_{\hat{f}_{a,i}(q)}\Phi(F, x, U))_b = (\sum_i u_{b,i}(g_{b,i}(x^1), \dots, g_{b,i}(x^n)), 0)$$

with $a, b = 1, \dots, M$, with arbitrary coefficients $\lambda_{a,i,q}$ and where $g_{a,i}(x) = \sum_q \lambda_{a,i,q} q^\perp e^{i\langle q, x\rangle}$ are arbitrary vectors in $\mathcal{F}^d$. Now note that for any $a = 1, \dots, M$ the constraint $|u_a|^2 = 1$

imply that there exists $i = 1, \dots, 2n$ such that $u_{a,i} \neq 0$. This allows to conclude that in the image of $D\Phi(F, x, U)$ belong all the vectors $((A_{h_1}(x), \rho_1), \dots, (A_{h_M}(x), \rho_M))$ for an arbitrary family $\{h_a \in \mathcal{F}_d\}_{a=1,\dots,M}$ and $\rho_1, .., \rho_M \in \mathbb{R}$. Now we use the assumption that for any $x \in G$ we have vector fields $\sigma_1, \dots, \sigma_{2n}$ such that $\{A_{\sigma_i}(x)\}_{i=1,\dots,2n}$ span all $\mathbb{R}^{2n}$. This is enough to conclude that for every (F, x, U) we have $\mathrm{Im}(D\Phi(F, x, U)) = (\mathbb{R}^{2n} \times \mathbb{R})^M$.

Now, by using Theorem 11 we deduce that for Lebesgue-almost every $F \in \mathcal{F}_d^{2nM}$ the set of configurations $x \in G$ and auxiliary vectors $U \in (\mathbb{R}^{2n})^M$ such that $\Phi(F, x, U) = 0$ is empty. This in turn implies that for almost every realization of Fourier coefficients the $2nM$ vector fields $\{A_{f_{a,i}}\}_{a=1,\dots,M, i=1,\dots,2n}$ span $\mathbb{R}^{2n}$ in each point $x \in G$ since for every $x \in G$ at least one of the combinations $\sum_{i=1}^{2n} u_{1,i} A_{f_{1,i}}(x), \dots, \sum_{i=1}^{2n} u_{M,i} A_{f_{M,i}}(x)$ is always different from zero for any possible choice of $u_{a,i}$. We just proved that

Theorem 13 *Assume that for every $x \in G$ there exists vector-fields $\sigma_1, \dots, \sigma_{2n} \in \mathcal{F}_d$ such that* $\mathrm{Span}\{A_{\sigma_i}(x)\}_{i=1,\dots,2n} = \mathbb{R}^{2n}$. *Then for any $M2n$ and for Lebesgue almost every realization of $2nM$ vector fields $\{f_{a,i} \in \mathbb{F}_d\}_{a=1,\dots,M, i=1,\dots,2n}$, the family $\{A_{f_{a,i}}\}_{a=1,\dots,M, i=1,\dots,2n}$ spans $\mathbb{R}^{2n}$ in all the points $x \in G$.*

Note that this theorem allows us to obtain a result valid in every point for a generic set of vector fields form a construction of a set of vector fields specific for each point, which is a lot easier to do.

For every $\delta > 0$ let us now define the open set $G_\delta = \{x \in \mathbb{T}^{2n} : \min_{i \neq j} |x^i - x^j| > \delta\} \subset \mathbb{T}^{2n}$ of points δ-uniformly away from the diagonals.

A simple construction gives that for each $\delta > 0$ there exist two smooth divergence-free vector fields $g_1(x)$ and $g_2(x)$ with compact support inside the ball $B(0, \delta/2)$ and such that $g_1(0) = (1, 0)$ and $g_2(0) = (0, 1)$ (it is sufficient to use fields of the form $g(x) = \varphi(|x - x_0|^2)(x - x_0)^\perp$ with suitable $x_0 \in \mathbb{R}^2$ and smooth scalar compact support function φ). In such a way, for any fixed point $\hat{x} \in G_\delta$ we can obtain $2n$ vector fields $f_1, \dots, f_{2n}$ of the form

$$f_{2k-1}(x) = g_1(x - \hat{x}_k), f_{2k}(x) = g_2(x - \hat{x}_k), k = 1, \dots, n$$

such that $\{A_{f_i}(\hat{x})\}_{i=1,\dots,2n}$ is the canonical basis of $\mathbb{R}^{2n}$. A difficulty stems from the fact that these fields do not necessarily belong to $\mathcal{F}_d$ for some d. We need then to approximate the functions g_1 and g_2 by elements of $\mathcal{F}_d$ for d large enough. Fix $\varepsilon > 0$ small enough, by density of trigonometric polynomials, there exists $d > 0$ and $\widetilde{g_1}, \widetilde{g_2} \in \mathcal{F}_d$ such that $\sup_{x \in \mathbb{T}^2} |g_i(x) - \tilde{g}_i(x)| < \varepsilon$. Note that the functions

$$\tilde{f}_{2k-1}(x) = \tilde{g}_1(x - \hat{x}_k), \tilde{f}_{2k}(x) = \tilde{g}_2(x - \hat{x}_k), k = 1, \dots, n$$

belong to $\mathcal{F}_d$ for any $\hat{x} \in \mathbb{T}^2$ and that, for example,

$$|A_{\tilde{f}_1(\hat{x})} - (1, 0, \dots, 0)| \leqslant C\varepsilon$$

where the constant does not depend on the parameters of the problem. Then for ε small enough, the family $\{A_{\tilde{f}_i(\hat{x})}\}_{i=1,\dots,2n}$ spans all $\mathbb{R}^{2n}$. The value of d depends only on ε and δ but not on $\hat{x} \in G_\delta$. This leads us to the following result.

Lemma 14 *For each $\delta > 0$ there exists $d \geqslant 1$ such that for every $x \in G_\delta$ we can find $2n$ vector fields $f_1, \dots, f_{2n} \in \mathcal{F}_d$ with the property that* $\mathrm{span}\{A_{f_i(x)}\} = \mathbb{R}^{2n}$.

An easy implication is then

Corollary 15 *For every $\delta > 0$ and $d > d_0(\delta)$, almost every realization of $2nM$ vector fields $\{f_{a,i} \in \mathcal{F}_d\}_{a=1,\dots,M,i=1,\dots,2n}$ is such that* $\mathrm{span}\{A_{f_{a,i}}(x)\}_{a=1,\dots,M,i=1,\dots,2n} = \mathbb{R}^{2n}$ *for all $x \in G_\delta$.*

By approximation of C^∞ vector fields by elements in $\mathcal{F}_d$ we can conclude that also the set $Q_\delta \subset (C^\infty(\mathbb{T}^2;\mathbb{R}^2))^{2nM}$ of $2nM$ vector fields $\{f_{a,i}\}_{a=1,\dots,M,i=1,\dots,2n}$ such that for all $x \in G_\delta$ $\mathrm{span}\{A_{f_{a,i}}(x)\}_{a=1,\dots,M,i=1,\dots,2n} = \mathbb{R}^{2n}$ is dense in $(C^\infty)^{2nM}$.

Let us prove that Q_δ contains an open dense subset. For any compact $K \subset \mathbb{T}^2$ define Q_K as the subset of $(C^\infty(\mathbb{T}^2;\mathbb{R}^2))^{2nM}$ which spans the full tangent space in every point of K.

We first prove that the sets Q_K are open: indeed assume that there exists a sequence $(f_{i,a}^{(k)}) \in Q_K^c$ such that $f^{(k)}$ converge to a point f in Q_K. For each $f^{(k)}$ there exists a point $x^{(k)} \in K$ for which $\mathrm{span}(A_{f_{i,a}^{(k)}}(x^{(k)})) \neq \mathbb{R}^{2n}$. By compactness of K we can extract a subsequence, still denoted by $(x^k)_{k\geq 1}$ which converges to $x \in K$. Then by uniform convergence of $f^{(k)}$ to f we deduce that we also have $\mathrm{span}(A_{f_{i,a}}(x)) \neq \mathbb{R}^{2n}$ which is in contradiction with the fact that $f \in Q_K$.

Then observe that for any $0 < \delta' < \delta$ there exists a compact K such that $G_\delta \subset K \subset G_{\delta'}$ and then that $Q_{\delta'} \subset Q_K \subset Q_\delta$. The set $Q_{\delta'}$ is dense and contained in an open set Q_K which proves that the interior of Q_δ is both open and dense, that is a residual set (or co-meagre).

At this point, by countable intersection, we get that $Q = \cap_k Q_{1/k}$ is also residual and its elements are exactly the vector fields such that $\mathrm{span}\{A_{f_{a,i}}(x)\} = \mathbb{R}^{2n}$ in every point of Γ^c. We have then proved Theorem 10.

A Remarks on hypoellipticity

We want to clarify the role of the nondegeneracy condition of the n-point motion assumed in section 3.1. Let us recall the following theorem. See for instance [17], Theorem 2.3.2.

Theorem 16 *Consider the stochastic equation in Stratonovich form in $\mathbb{R}^m$*

$$X_t = x_0 + \sum_{j=1}^{N} \int_0^t A_j(X_s) \circ dW_s^j + \int_0^t A_0(X_s)ds$$

with infinitely differentiable coefficients with bounded derivatives of all order. Assume the following Hörmander's condition at point x_0: the vector space spanned by the vector fields

$$A_1, ..., A_N, \qquad [A_i, A_j], 0 \le i, j \le N, \qquad [A_i, [A_j, A_k]], 0 \le i, j, k \le N, ...$$

at point x_0 is $\mathbb{R}^m$. Then, for every $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure.

When the vector fields $A_1, ..., A_N$ themselves span $\mathbb{R}^m$, there is a simpler criterium, due to [3]. We recall a simplified version of Theorem 2.3.1 from [17]. Denote by $A(x)$ the $m \times N$ matrix with $A_1(x), ..., A_N(x)$ as columns and by $\sigma(x)$ the $m \times m$ matrix $A(x)A(x)^T$.

Theorem 17 *Let $(X_t)_{t\ge 0}$ be a solution of the Itô equation in $\mathbb{R}^m$*

$$X_t = x_0 + \sum_{j=1}^{N} \int_0^t A_j(X_s) dW_s^j + \int_0^t A_0(X_s) ds \tag{15}$$

with globally Lipschitz coefficients. Assume

$$P\left(\int_0^t 1_{\{\det \sigma(X_s) \neq 0\}} ds > 0\right) = 1$$

for all $t > 0$. Then, for every $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure.

Corollary 18 *Let $(X_t)_{t\ge 0}$ be a solution of the Itô equation (15), with globally Lipschitz coefficients. If $A_1(x), ..., A_N(x)$ generate $\mathbb{R}^m$ at $x = x_0$, then, for every $t > 0$, the law of X_t is absolutely continuous with respect to the Lebesgue measure.*

Proof. Since the fields are continuous, $A_1(x), ..., A_N(x)$ generate $\mathbb{R}^m$ at all points of a neighbor $\mathcal{U}$ of x_0. The solution $(X_t)_{t\ge 0}$ has continuous paths, thus belongs to $\mathcal{U}$ at least over a small random time interval $[0, \tau]$, $P(\tau > 0) = 1$. On $\mathcal{U}$ we have $\det \sigma(x) \neq 0$, hence the assumption of the theorem is satisfied. The proof is complete. ■

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